A LOOK AT THE ADAPTIVE HOPF OSCILLATORS: AN OVERVIEW (WEB PLATFORM UPGRADE)

**Maria Vasileva, Vesselin Kyurkchiev, Anna Malinova,
Anton Iliev, Asen Rahnev, Nikolay Kyurkchiev**

Abstract. *In this article we demonstrate some specialized modules for investigating the dynamics of some generalized Hopf oscillators, an integral part of a planned much more general Web-based application for scientific computing. We also study some new hypothetical adaptive Hopf-like oscillators. Numerical examples, illustrating our results using CAS MATHEMATICA are given.*

Key words: Adaptive Hopf oscillator, Three-state Hopf oscillator, Four-state Hopf oscillator, Modified Hopf oscillator.

Mathematics Subject Classification: 65L07, 34A34

1. Two–state Hopf oscillator

The Hopf oscillator is a nonlinear oscillator described by the following ordinary differential equations (see for example [1]):

$$\begin{cases} \dfrac{dx}{dt} &= (a_1 - (x^2+y^2))x - dy + k(a\sin(a_2 t + a_3)) \\ \dfrac{dy}{dt} &= (a_1 - (x^2+y^2))y + dx \end{cases} \tag{1}$$

where d is a resonance constant, a_1 is a constant that controls the limit cycle radius, and k is a coupling constant.

The input signal is $a\sin(a_2 t + a_3)$, where a is the amplitude of the sinusoid, a_2 is the external forcing frequency, and a_3 is the phase of the input sinusoid. This nonlinear oscillator is used as the building block for the subsequent adaptive oscillator systems. Since this system is not adaptive, the frequency has a single peak.

Some simulations.

I. For given $a_1 = 0.001$, $d = 0.31$, $k = 0.02$, $a = 0.1$, $a_2 = 0.35$, $a_3 = 0.19$ the simulations on the system (1) for $x_0 = 0.9$; $y_0 = 0.8$ are depicted on Figure 1.

I.1 For given $a_1 = 0.001$, $d = 0.03$, $k = 0.013$, $a = 0.2$, $a_2 = 0.4$, $a_3 = 0.2$ the simulations on the system (1) for $x_0 = 0.9$; $y_0 = 0.9$ are depicted on Figure 2.

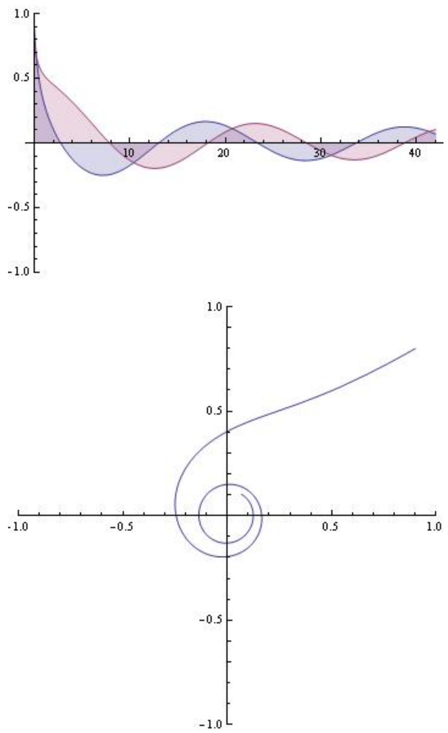

Figure 1. a) The solutions of differential system; b) Phase portrait; (example I).

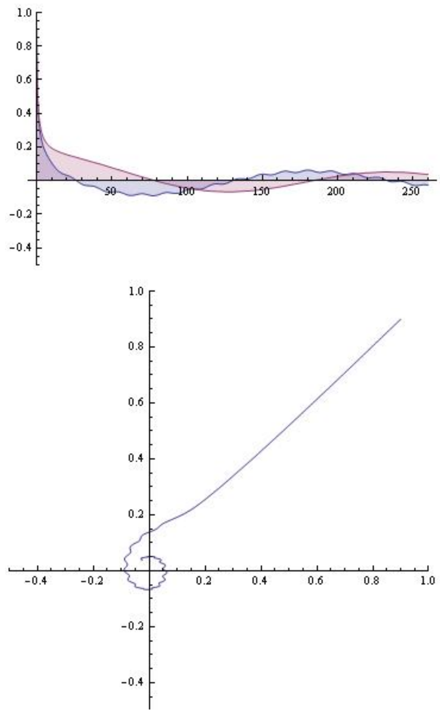

Figure 2. a) The solutions of differential system; b) Phase portrait; (example I.1).

2. A modified two–state Hopf oscillator

We consider the following new extension of the model (1):

$$\begin{cases} \dfrac{dx}{dt} = (a_1 - (x^2 + y^2))x - d(t)y + k(t)(a\sin(a_2 t + a_3)) \\ \dfrac{dy}{dt} = (a_1 - (x^2 + y^2))y + d(t)x \end{cases} \tag{2}$$

$$d(t) = \sum_{i=0}^{l} \alpha_i t^i; \quad k(t) = \sum_{i=0}^{n} \beta_i t^i.$$

Some simulations.

II. For given $a_1 = 0.001$, $a = 0.2$, $a_2 = 0.4$, $a_3 = 0.2$ and $d(t) = 0.0001 + 0.015t - 0.000001t^2$, $k(t) = 0.002 - 0.000001t + 0.00000001t^2$ the simulations on the system (2) for $x_0 = 0.9$; $y_0 = 0.9$ are depicted on Figure 3a and Figure 3b. For the phase portrait see Figure 3c.

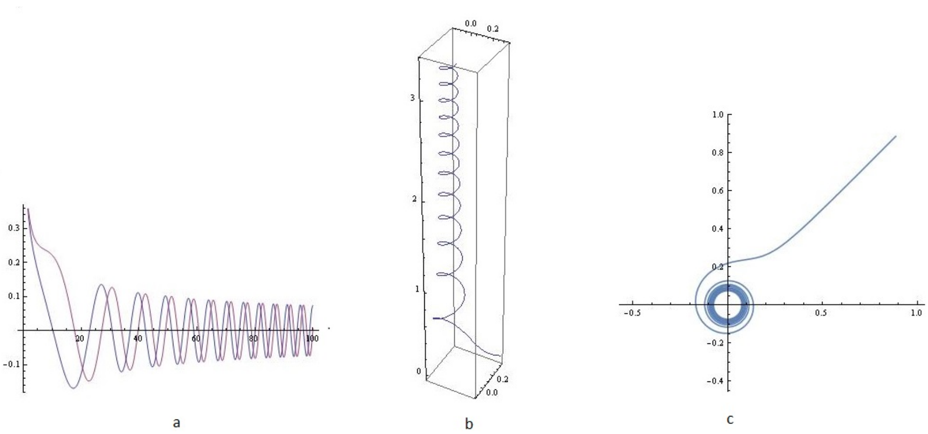

Figure 3. a) The solutions of differential system;
b) ParametricPlot3D[...] in CAS Mathematica for $x(t), y(t)$; (example II);
c) The phase portrait (example II).

II.1 For given $a_1 = 0.01$, $a = 0.1$, $a_2 = 0.9$, $a_3 = 0.7$ and $d(t) = 0.0001 + 0.015t - 0.000001t^2$, $k(t) = 0.002 - 0.000001t + 0.00000001t^2$ the simulations on the system (2) for $x_0 = 0.2$; $y_0 = 0.1$ are depicted on Figure 4a and Figure 4b. For the phase portrait see Figure 4c.

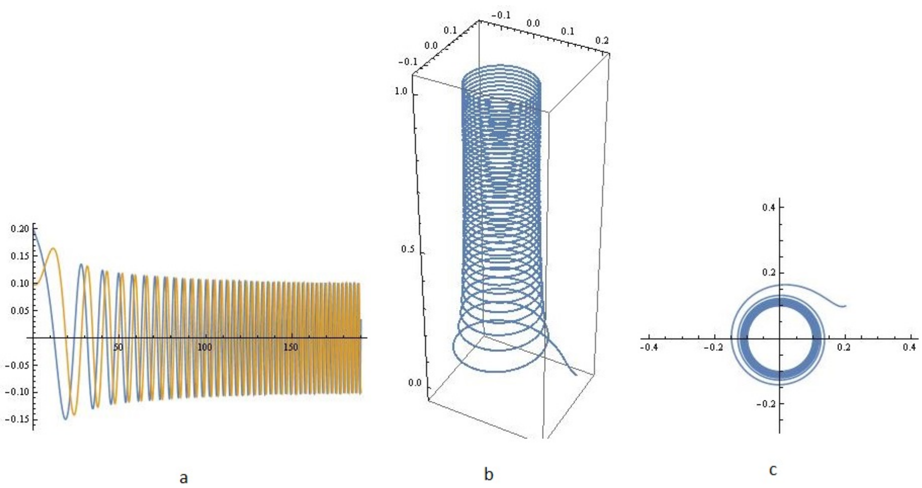

Figure 4. a) The solutions of differential system;
b) ParametricPlot3D[...] in CAS Mathematica for $x(t), y(t)$; (example II.1);
c) The phase portrait (example II.1).

II.2 For given $a_1 = 0.02$, $a = 0.2$, $a_2 = 0.6$, $a_3 = 0.5$ and $d(t) = 0.0001 + 0.025t - 0.000013t^2 + 0.000000001t^3$, $k(t) = 0.005 - 0.00001t + 0.0000001t^2 - 0.00000001t^3$ the simulations on the system (2) for $x_0 = 0.3$; $y_0 = 0.1$ are depicted on Figure 5a and Figure 5b. For the phase portrait see Figure 5c.

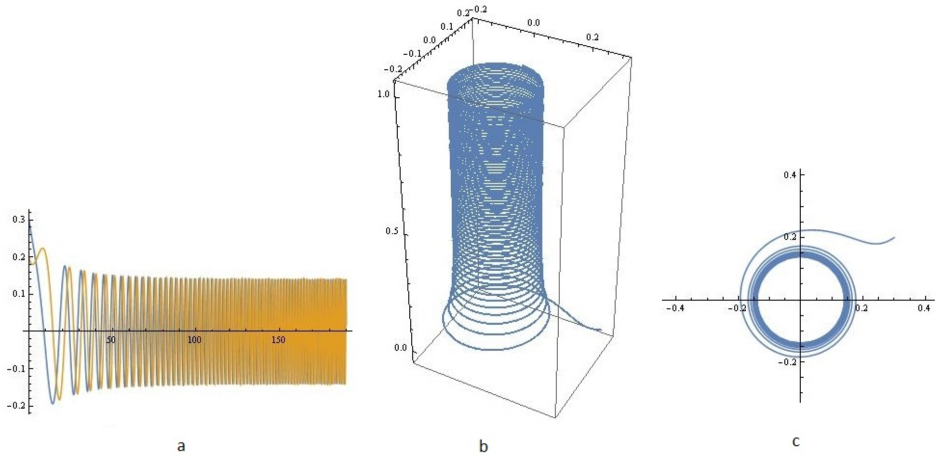

Figure 5. a) The solutions of differential system;
b) ParametricPlot3D[...] in CAS Mathematica for $x(t), y(t)$; (example II.2); The phase portrait (example II.2).

3. Three–state adaptive Hopf oscillator

The three state Hopf oscillator is defined as:

$$\begin{cases} \dfrac{dx}{dt} &= (a_1 - (x^2 + y^2))x - dy + k(a\sin(a_2 t + a_3)) \\ \dfrac{dy}{dt} &= (a_1 - (x^2 + y^2))y + dx \\ \dfrac{dd}{dt} &= -k(a\sin(a_2 t + a_3))y \end{cases} \tag{3}$$

It should be noted that the external input signal is injected into both the x and the d states.

Some simulations.

III. For given $k = 0.013$, $a_1 = 0.001$, $a = 0.2$, $a_2 = 0.4$, $a_3 = 0.2$ the simulations on the system (3) for $x_0 = 0.9$; $y_0 = 0.8$, $d_0 = 0.16$ are depicted on Figure 6a. Using operator *ParametricPlot3D*[...] in CAS Mathematica for component of solution – $x(t), y(t)$ see Figure 6b.

III.1 For given $k = 0.014, a_1 = 0.0009, a = 1, a_2 = 0.05, a_3 = 0.01$ the simulations on the system (3) for $x_0 = 0.5$; $y_0 = 0.4$, $d_0 = 0.16$ are depicted on Figure 7a. Using operator $ParametricPlot3D[...]$ in CAS Mathematica for component of solution – $x(t), y(t)$ see Figure 7b.

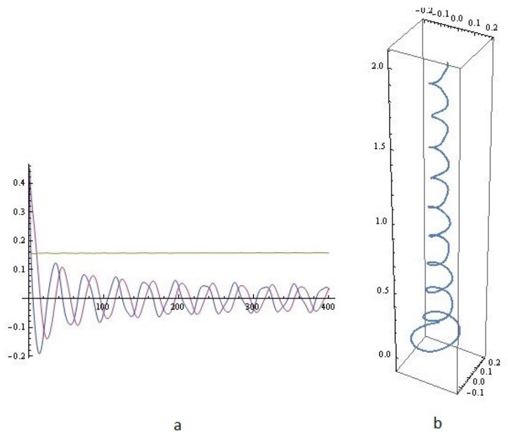

Figure 6. a) The solutions of differential system; (example III); b) $ParametricPlot3D[...]$; (example III).

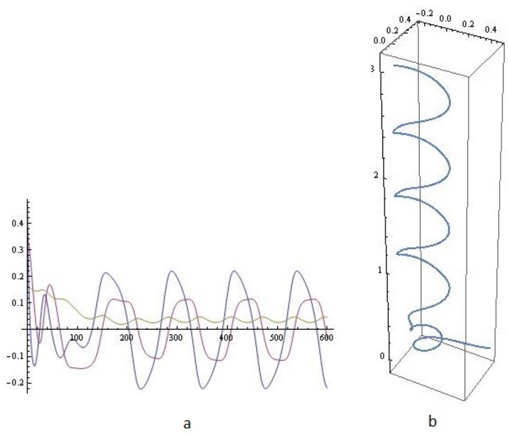

Figure 7. a) The solutions of differential system; (example III.1); b) $ParametricPlot3D[...]$; (example III.1).

4. Four–state adaptive Hopf oscillator

The four–state Hopf oscillator is defined as (see for example [2]):

$$\begin{cases} \dfrac{dx}{dt} &= (a_1 - (x^2 + y^2))x - dy + k(a\sin(a_2 t + a_3) - k_1 b) \\ \dfrac{dy}{dt} &= (a_1 - (x^2 + y^2))y + dx \\ \dfrac{dd}{dt} &= -k(a\sin(a_2 t + a_3) - k_1 b)y \\ \dfrac{db}{dt} &= a_4(a\sin(a_2 t + a_3) - k_1 b)x \end{cases} \tag{4}$$

Here a_4 is a coupled constant. A circuit implementation of the system (4) was designed, fabricated and tested is given in [2].

Some simulations.

IV. For given $k = 0.013$, $k_1 = 0.15$, $a_1 = 0.001$, $a = 0.2$, $a_2 = 0.4$, $a_3 = 0.2$, $a_4 = 0.3$ the simulations on the system (4) for $x_0 = 0.9$; $y_0 = 0.8$, $d_0 = 0.13$, $b_0 = 0.12$ are depicted on Figure 8.

IV.1 For given $k = 0.014$, $k_1 = 0.01$, $a_1 = 0.0009$, $a = 1$, $a_2 = 0.04$, $a_3 = 0.02$, $a_4 = 0.001$ the simulations on the system (4) for $x_0 = 0.9$; $y_0 = 0.8$, $d_0 = 0.13$, $b_0 = 0.12$ are depicted on Figure 9.

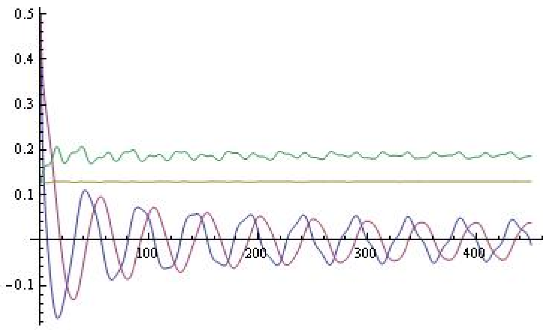

Figure 8. The solutions of differential system; (example IV).

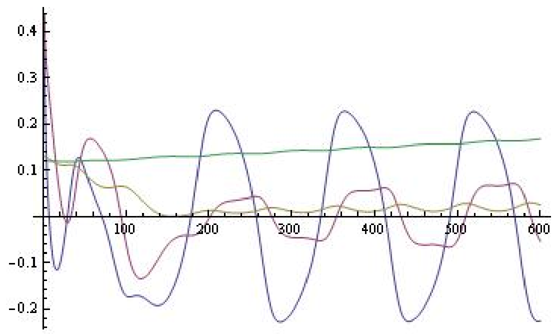

Figure 9. The solutions of differential system; (example IV.1).

5. A new modified four–state adaptive Hopf oscillator

We define the following modification of the four–state adaptive Hopf oscillator:

$$\begin{cases} \dfrac{dx}{dt} & = & (a_1 - (x^2 + y^2))x - dy + k(a\sin(a_2 t + a_3) - k_1 b) \\ \dfrac{dy}{dt} & = & (a_1 - (x^2 + y^2))y + dx \\ \dfrac{dd}{dt} & = & -k(a\sin(a_2 t + a_3) - k_1 b)y \\ \dfrac{db}{dt} & = & a_4(t)(a\sin(a_2 t + a_3) - k_1 b)x \end{cases} \tag{5}$$

where $a_4(t) = \displaystyle\sum_{i=0}^{n} \gamma_i t^i$.

Remark. Performing a rigorous local analysis on system (5) follows the idea discussed in [1] and will be omitted here.

Some simulations.

V. For given $k = 0.014$, $k_1 = 0.01$, $a_1 = 0.0009$, $a = 1$, $a_2 = 0.4$, $a_3 = 0.2$ and $a_4(t) = 0.0001 + 0.0005t - 0.0000001t^2 + 0.000000001t^3$ the simulations on the system (5) for $x_0 = 0.9$; $y_0 = 0.8$, $d_0 = 0.13$, $b_0 = 0.12$ are depicted on Figure 10a and Figure 10b.

V.1 For given $k = 0.015$, $k_1 = 0.04$, $a_1 = 0.007$, $a = 1.1$, $a_2 = 0.07$, $a_3 = 0.05$ and $a_4(t) = 0.01 - 0.0003t + 0.0000002t^2 - 0.000000003t^3$ the simulations on the system (5) for $x_0 = 0.9$; $y_0 = 0.8$, $d_0 = 0.13$, $b_0 = 0.12$ are depicted on Figure 11a–11b.

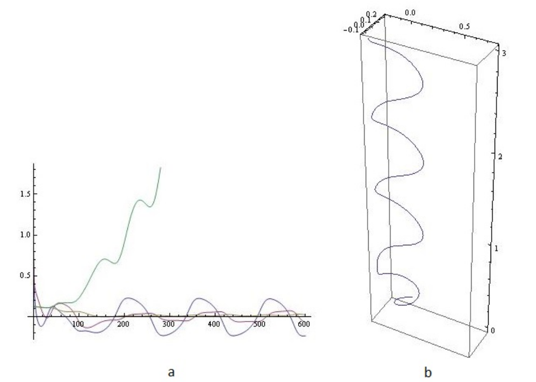

Figure 10. a) The solutions of differential system; (example V); b) ParametricPlot3D[...]; (example V).

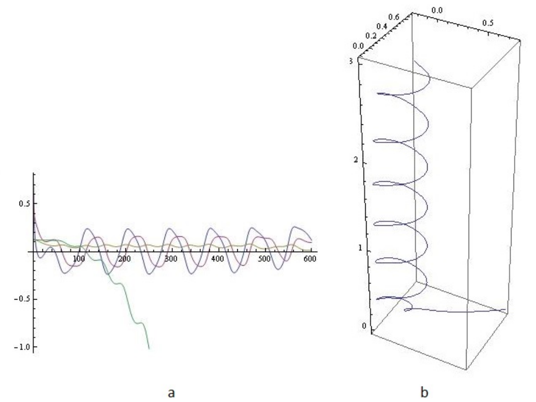

Figure 11. a) The solutions of differential system; (example V.1); b) ParametricPlot3D[...]; (example V.1).

6. Concluding Remarks

In addition to the possibilities described in detail in [3]–[15], which our Web–application provides, in this paper we demonstrated some modules with which the planned Web–platform is upgraded, in the following directions:

– dynamics of the two–state Hopf oscillator;

– dynamics of the new modified two–state Hopf oscillator;

– dynamics of the three and four–state adaptive Hopf oscillators;

– dynamics of the new modified four–state adaptive Hopf oscillator with a suitable, user-fixed "coupled function".

The following three examples using the new model (5) illustrate interesting dynamics typical of this type of oscillators. We hope that these results will be of interest to specialists working in this scientific direction.

We presented only a small part of the platform's capabilities.

The Web platform requires the user to enter specialized data needed to run simulations, especially for the new models of type (2) and (5) (coefficients of the polynomial $d(t)$, $k(t)$ and $a_4(t)$).

Coupled nonlinear oscillators are abundant in biology, physics, and chemical reaction systems [16]–[18].

Some popular techniques for deriving the phase dynamics of coupled oscillators can be found in [19].

We will explicitly note that the theoretical apparatus for studying the circuit implementation (design, fabricating, etc.) of the considered differential models is extremely complex and requires serious consideration before being adapted for its possible inclusion in our planned Web–based platform for scientific calculations. Of course, such an analysis is imperative and remains our top priority for future development.

Acknowledgment

This study is financed by the project No FP23-FMI-002 “Intelligent software tools and applications in research in Mathematics, Informatics, and Pedagogy of Education” of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

References

[1] A. Nayfeh, B. Balachandran, *Applied Nonlinear Dynamics: analytical, computational, snd experimental methods*, John Wiley and Sons, 2008.

[2] X. Li, M. Shoudat, S. Kennedy, C. Fendley, R. Dean, A. Beal, F. Perkins, A four–state adaptive Hopf oscillator, *PLoS ONE*, 2001, 16 (3).

[3] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, 2023, 16 (6), 297.

[4] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On some extended oscillator models: a technique for simulating and studying their dynamics*, Plovdiv University Press: Plovdiv, 2022, ISBN: 978-619-7663-13-6.

[5] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On the Lienard System with Some “Corrections of Polynomial-type”: Number of Limit Cycles, Simulations and Possible Applications. Part II.*, Plovdiv, Plovdiv University Press, 2022, ISBN: 978-619-7663-41-9.

[6] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, *Applications of Some Orthogonal Polynomials and Associated Polynomials of Higher Kind*, Plovdiv, Plovdiv University Press, 2023, ISBN: 978-619-7663-72-3.

[7] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms*, 2022, 15 (12), 462.

[8] N. Kyurkchiev, T. Zaevski, On a hypothetical oscillator: investigations in the light of the Melnikov's approach, some simulations, *International Journal of Differential Equations and Applications*, Vol. 22, No. 1 (2023), 67–79.

[9] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, Some investigations and simulations on the planar Rayleigh–Lienard differential system (Web Platform upgrade), *International Journal of Differential Equations and Applications*, 2023, 22, No. 1, 81–102.

[10] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, Dynamics and simulations of some generalized Rayleigh systems (Web Platform upgrade), *International Journal of Differential Equations and Applications*, 2023, 22, No. 1, 103–121.

[11] A. Golev, V. Arnaudova, Dynamics of a generalized hypothetical Rayleigh–Duffing–like oscillator: investigations in the light of Melnikov's approach, simulations (Web–Platform upgrade), *International Journal of Differential Equations and Applications*, 2023, 22, No. 1, 123–136.

[12] A. Malinova, N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, A look at the Duffing system with periodic parametric excitation: an overview (Web–Platform upgrade), *International Journal of Differential Equations and Applications*, 2023, 22, No. 1, 137–154.

[13] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, On a Duffing–like oscillators with "transfer excitation function (TEF)": some hypothetical models (Web–Platform upgrade), *Communications in Applied Analysis*, 2023.

[14] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, A look at the quasi–periodic oscillations in a self–excited Mathieu oscillator: an overview (Web–Platform upgrade), *Communications in Applied Analysis*, 2023.

[15] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, On a Mathieu–like oscillator with "transfer excitation function" (Web–Platform upgrade), *Communications in Applied Analysis*, 2023.

[16] A. Pikosky, et al., *Synchronization, a Universal Concept in Nonlinear Sciences*, Cambridge Univ. Press, 2001.

[17] A. Winfree, *The Geometry of Biological Times*, Springer, 1980.

[18] Y. Kuramoto, *Chemical Oscillators, Waves and Turbulence*, Springer-Verlag, Berlin, 1984.

[19] B. Pietras, A. Daffertshofer, Network dynamics of coupled oscillators and phase reduction techniques, *Physics Reports*, 2019, 819, 1–105.

Maria Vasileva[1], Vesselin Kyurkchiev[2], Anna Malinova[3],
Anton Iliev[4,*], Asen Rahnev[5], Nikolay Kyurkchiev[6]
[1,2,3,4,5,6] Faculty of Mathematics and Informatics,
University of Plovdiv Paisii Hilendarski,
24, Tzar Asen Str., 4000 Plovdiv, Bulgaria,
[4,6] Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria
Corresponding author: aii@uni-plovdiv.bg