YAMABE SOLITONS AND YAMABE ALMOST SOLITONS WITH VERTICAL POTENTIAL ON SOME SPECIAL TYPES OF ALMOST CONTACT COMPLEX RIEMANNIAN MANIFOLDS

Mancho Manev

Abstract. *We summarize our results on Yamabe solitons and Yamabe almost solitons considered on almost contact complex Riemannian manifolds, known also as almost contact B-metric manifolds. These manifolds are endowed with a pair of mutually associated pseudo-Riemannian metrics with respect to the almost contact structure. Each of these metrics is specialized as a Yamabe (almost) soliton with a vertical potential, i.e. collinear to the Reeb vector field. The resulting manifolds are then investigated in three important cases with geometric significance. The first is when the manifold is cosymplectic, i.e. with parallel structure tensors. The second case is of such a manifold of Sasaki-like type, i.e. its complex cone is a holomorphic complex Riemannian manifold (also called a Kähler–Norden manifold). The third case is when the soliton potential is torse-forming, i.e. it satisfies a certain recurrence condition for its covariant derivative with respect to the Levi-Civita connection of the corresponding metric. The studied solitons are characterized. Explicit examples are commented, and the properties obtained in the theoretical part are confirmed.*

Key words: Yamabe soliton, almost contact B-metric manifold, almost contact complex Riemannian manifold, Sasaki-like manifold, torse-forming vector field

Mathematics Subject Classification: 53C25, 53D15, 53C50, 53C44, 53D35, 70G45.

1. Introduction

The notion of the Yamabe flow is introduced in [1, 2] by R. S. Hamilton to construct metrics with constant scalar curvature $\tau(t)$ corresponding to a time-dependent family of (pseudo-)Riemannian metrics $g(t)$ on a smooth manifold $\mathcal{M}$. The metric $g(t)$ is said to evolve by *Yamabe flow* if

$g(t)$ satisfies the following evolution equation

$$\frac{\partial}{\partial t}g(t) = -\tau(t)g(t), \qquad g(0) = g_0.$$

A self-similar solution of the Yamabe flow on $(\mathcal{M}, g)$ is called a *Yamabe soliton* and is determined by the following equation

$$\frac{1}{2}\mathcal{L}_\vartheta g = (\tau - \lambda)g, \tag{1}$$

where $\mathcal{L}_\vartheta g$ denotes the Lie derivative of g along the vector field ϑ called the soliton potential, and λ is the soliton constant (e.g. [3]). Briefly, we denote this soliton by $(g; \vartheta, \lambda)$. In the case that λ is a differential function on $\mathcal{M}$, the solution is called an *Yamabe almost soliton*.

Many authors have studied Yamabe (almost) solitons on different types of manifolds in the recent years (see e.g. [4, 5, 6, 7, 8, 9, 10]). The investigations of this kind of flows and the corresponding (almost) solitons cause an interest in mathematical physics because the Yamabe flow corresponds to fast diffusion of the porous medium equation [12].

The study of Yamabe solitons on almost contact complex Riemannian manifolds (abbreviated accR manifolds), there called almost contact B-metric manifolds, was started by the author with [9]. The geometry of these manifolds is largely determined by the presence of a pair of B-metrics that are related each other by the almost contact structure. Two of the simplest types of the considered manifolds are studied there, namely cosymplectic and Sasaki-like, introduced in [15].

In [10], the author continues this study for Yamabe solitons on accR manifolds with vertical torse-forming potential derived by contact conformal transformation of general type.

The study continues in [11] with the use of a condition for Yamabe almost soliton for each of the B-metrics in the case of Sasaki-like manifolds.

The present paper gives a review on the latest results on this special types metrics for the manifolds under interest.

2. accR manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an accR manifold, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ, ξ, η) and a pair of pseudo-Riemannian metrics g and $\tilde{g}$ of signature $(n+1, n)$, such that [13]

$$\varphi\xi = 0, \quad \varphi^2 = -\iota + \eta\otimes\xi, \quad \eta\circ\varphi = 0, \quad \eta(\xi) = 1,$$
$$g(X,Y) = -g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \quad \tilde{g}(X,Y) = g(X, \varphi Y) + \eta(X)\eta(Y),$$

where ι stands for the identity transformation on the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M. Here and further, X, Y, Z will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space T_pM of M at an arbitrary point p in M.

The fundamental tensor F of type (0,3) on $(\mathcal{M}, \varphi, \xi, \eta, g)$ is defined by $F(X,Y,Z) = g\big((\nabla_X\varphi)Y, Z\big)$, where ∇ is the Levi-Civita connection of g, and the following Lee forms are associated with it:

$$\theta = g^{ij}F(E_i, E_j, \cdot), \qquad \theta^* = g^{ij}F(E_i, \varphi E_j, \cdot), \qquad \omega = F(\xi, \xi, \cdot),$$

where g^{ij} are the components of the inverse matrix of g with respect to a basis $\{E_i; \xi\}$ $(i = 1, 2, \ldots, 2n)$ of T_pM.

A classification of these manifolds in terms of F is given in [13]. This classification includes eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\ldots$, $\mathcal{F}_{11}$, intersected in the class $\mathcal{F}_0$ of cosymplectic B-metric manifolds defined by condition $F = 0$.

A contact conformal transformation of general type is introduced in [14] as follows

$$\begin{aligned} \bar{\xi} &= e^{-w}\xi, & \bar{g} &= e^{2u}\cos 2v\, g + e^{2u}\sin 2v\, \tilde{g} \\ \bar{\eta} &= e^{w}\eta, & &+ \big(e^{2w} - e^{2u}\cos 2v - e^{2u}\sin 2v\big)\, \eta\otimes\eta, \end{aligned} \tag{2}$$

where u, v, w are differentiable functions on M. Such a transformation preserves the manifolds of the main classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_{11}$.

3. Yamabe solitons

As in [9], we say that the B-metric $\bar{g}$ with a scalar curvature $\bar{\tau}$ generates a Yamabe soliton with potential $\bar{\xi}$ and soliton constant $\bar{\sigma}$ on a conformal accR manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, if the following condition is satisfied

$$\frac{1}{2}\mathcal{L}_{\bar{\xi}}\bar{g} = (\bar{\tau} - \bar{\sigma})\bar{g}.$$

In [9] it was proved that an accR manifold which is cosymplectic can be transformed by (2) so that $\bar{g}$ is a Yamabe soliton with potential $\bar{\xi}$

and a soliton constant $\bar{\sigma}$ if and only if the functions (u, v, w) of the used transformation satisfy the conditions

$$\mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 0, \qquad \mathrm{d}w = \mathrm{d}w(\xi)\eta.$$

Moreover, the obtained Yamabe soliton has a constant scalar curvature $\bar{\tau} = \bar{\sigma}$ and the obtained accR manifold belongs to the subclass of $\mathcal{F}_1$ determined by:

$$\bar{\theta} = 2n\left\{\mathrm{d}u \circ \varphi - \mathrm{d}v \circ \varphi^2\right\}, \qquad \bar{\theta}^* = -2n\left\{\mathrm{d}u \circ \varphi^2 + \mathrm{d}v \circ \varphi\right\}.$$

If $(\mathcal{M}, \varphi, \xi, \eta, g)$ is Sasaki-like, then the condition $\nabla_X \xi = -\varphi X$ is met. These manifolds are determined in [15] in terms of F by

$$F(X, Y, Z) = g(\varphi X, \varphi Y)\eta(Z) + g(\varphi X, \varphi Z)\eta(Y).$$

Then, we have proved that an accR manifold that is Sasaki-like can be transformed by (2) so that $\bar{g}$ is a Yamabe soliton with potential $\bar{\xi}$ and a soliton constant $\bar{\sigma}$ if and only if the functions (u, v, w) of the used transformation satisfy the conditions

$$\mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 1, \qquad \mathrm{d}w = \mathrm{d}w(\xi)\eta.$$

Moreover, the obtained Yamabe soliton has a constant scalar curvature $\bar{\tau} = \bar{\sigma}$ and the obtained manifold belongs to a subclass of the main class $\mathcal{F}_1$ determined by:

$$\bar{\theta} = 2n\left\{\mathrm{d}u \circ \varphi - \mathrm{d}v \circ \varphi^2\right\}, \qquad \bar{\theta}^* = 2n\left\{\mathrm{d}u - \mathrm{d}v \circ \varphi\right\}.$$

4. Yamabe solitons, contact conformal transformations and torse-forming vector fields

Further, we consider a torse-forming vector field ϑ on an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$. A vector field ϑ on $(\mathcal{M}, g)$ is called *torse-forming vector field* if it satisfies the following condition for arbitrary vector field $x \in \mathfrak{X}(\mathcal{M})$

$$\nabla_x \vartheta = f\, x + \gamma(x)\vartheta,$$

where f is a differentiable function and γ is a 1-form [18, 16]. The 1-form γ is called the *generating form* and the function f is called the *conformal* *scalar* of ϑ [17]. The essential example of a torse-forming ξ is when the manifold is of $\mathcal{F}_5$.

It is significant case when ϑ is a vertical vector field on $\mathcal{M}$, i.e. $\vartheta = k\,\xi$, where k is a nonzero function on $\mathcal{M}$ and obviously $k = \eta(\vartheta)$ holds true.

It was proved in [10] that an $\mathcal{F}_5$-manifold with a vertical torse-forming vector field ϑ can be transformed by an accR transformation so that $\bar{g}$ is a Yamabe soliton with potential ϑ and a soliton constant σ if and only if the functions (u, v, w) of the used transformation satisfy the conditions

$$\mathrm{d}u(\xi) = -\frac{f}{k}, \qquad \mathrm{d}v(\xi) = 0, \qquad \mathrm{d}w = \mathrm{d}w(\xi)\eta.$$

Moreover, the obtained Yamabe soliton has a constant scalar curvature $\bar{\tau} = \sigma$ and the obtained accR manifold belongs to the subclass of $\mathcal{F}_1$ determined by

$$\bar{\theta} = 2n\left\{\mathrm{d}u \circ \varphi + \mathrm{d}v\right\}, \qquad \bar{\theta}^* = -2n\left\{\mathrm{d}u \circ \varphi^2 + \mathrm{d}v \circ \varphi\right\}, \qquad \bar{\omega} = 0.$$

As a corollary, if (u, v) are a pair of φ-holomorphic functions then the transformed manifold belongs to the special class $\mathcal{F}_0$ of cosymplectic accR manifolds.

5. Pair of associated Yamabe almost solitons

Let us consider an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ with a pair of associated Yamabe almost solitons generated by the pair of B-metrics g and $\tilde{g}$, i.e. $(g; \vartheta, \lambda)$ and $(\tilde{g}; \tilde{\vartheta}, \tilde{\lambda})$, which are mutually associated by the (φ, ξ, η)-structure. Then, along with (1), the following identity also holds

$$\frac{1}{2}\mathcal{L}_{\tilde{\vartheta}}\tilde{g} = (\tilde{\tau} - \tilde{\lambda})\tilde{g},$$

where $\tilde{\vartheta}$ and $\tilde{\lambda}$ are the soliton potential and the soliton function, respectively, and $\tilde{\tau}$ is the scalar curvature of the manifold with respect to $\tilde{g}$. We suppose that the potentials ϑ and $\tilde{\vartheta}$ are vertical, i.e. there exists differentiable functions k and $\tilde{k}$ on $\mathcal{M}$, such that we have

$$\vartheta = k\xi, \qquad \tilde{\vartheta} = \tilde{k}\xi,$$

where $k(p) \neq 0$ and $\tilde{k}(p) \neq 0$ at every point p of M. Briefly, we denote these potentials by (ϑ, k) and $(\tilde{\vartheta}, \tilde{k})$.

5.1. The case when the underlying accR manifold is Sasaki-like

Obviously, Sasaki-like accR manifolds form a subclass of the class $\mathcal{F}_4$ of the Ganchev–Mihova–Gribachev classification. According to [11], there does not exist a Sasaki-like manifold $(\mathcal{M},\varphi,\xi,\eta,g)$ equipped with a g-generated Yamabe almost soliton having a vertical potential. The corresponding statement for $\tilde{g}$-generated Yamabe almost soliton with a vertical potential is also valid.

5.2. The case of a torse-forming vertical potential

Let an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g)$ be equipped with a Yamabe almost soliton $(g;\vartheta(f,k),\lambda)$, where ϑ is a vertical torse-forming potential.

Then the scalar curvature τ of this manifold is the sum of the conformal scalar f of ϑ and the soliton function λ, i.e. $\tau = f+\lambda$. As a corollary we have that the potential $\vartheta(f,k)$ of any Yamabe almost soliton $(g;\vartheta,\lambda)$ on $(\mathcal{M},\varphi,\xi,\eta,g)$ is a torqued vector field, i.e. $\gamma(\vartheta)=0$.

Similarly, the following analogous assertions are valid. Let an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g)$ be equipped with $(\tilde{g};\tilde{\vartheta}(\tilde{f},\tilde{k}),\tilde{\lambda})$, a Yamabe almost soliton with a vertical torse-forming potential $\tilde{\vartheta}$. Then the scalar curvature $\tilde{\tau}$ of this manifold is the sum of the conformal scalar $\tilde{f}$ of $\tilde{\vartheta}$ and the soliton function $\tilde{\lambda}$, i.e. $\tilde{\tau} = \tilde{f}+\tilde{\lambda}$. The potential $\tilde{\vartheta}(\tilde{f},\tilde{k})$ of any Yamabe almost soliton $(\tilde{g};\tilde{\vartheta},\tilde{\lambda})$ on $(\mathcal{M},\varphi,\xi,\eta,g)$ is a torqued vector field.

The relation between tha scalar curvatures in this case takes the form

$$\tilde{\tau} = -\tau^* - 2n(2n+1)h^2 - 4n\,\mathrm{d}h(\xi), \qquad h=\frac{f}{k}.$$

Finally, various explicit examples are given in [9, 10, 11] of the studied manifolds. They are constructed as a 5-dimensional Lie group, a hypersurface in $(2n+2)$-dimensional real space and the cone over a 2-dimensional complex space form with Norden metric. The computed characteristics for them support the obtained theoretical results.

Acknowledgments

This research was partially supported by project FP23-FMI-002 of the Scientific Research Fund, University of Plovdiv Paisii Hilendarski.

References

[1] R. Hamilton, The Ricci flow on surfaces, In *Mathematics and General Relativity* (Santa Cruz, CA, 1986) Contemp. Math., Vol. 71, Amer. Math. Soc. Providence, RI, 1988, 237–262.

[2] R. Hamilton, Lectures on geometric flows, *Unpublished manuscript*, 1989.

[3] E. Barbosa, E. Ribeiro Jr., On conformal solutions of the Yamabe flow, *Arch. Math.*, 2013, Vol. 101, 2013, 79–89.

[4] H. Cao, X. Sun, Y. Zhang, On the structure of gradient Yamabe solitons, *Math. Res. Lett.*, 2012, Vol. 19, No. 4, 767–774.

[5] B. Chen, S. Deshmukh, A note on Yamabe solitons, *Balk. J. Geom. Appl.*, 2018, Vol. 23, No. 1, 37–43.

[6] P. Daskalopoulos, N. Sesum, The classification of locally conformally flat Yamabe solitons, *Adv. Math.*, 2013, Vol. 240, 346–369.

[7] A. Ghosh, Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold, *Math. Slovaca*, 2020, Vol. 70, No. 1, 151–160.

[8] S. Roy, S. Dey, A. Bhattacharyya, Yamabe solitons on $(LCS)_n$-manifolds, *J. Dyn. Syst. Geom. Theor.*, 2020, Vol. 18, No. 2, 261–279.

[9] M. Manev, Yamabe solitons on conformal Sasaki-like almost contact B-metric manifolds, *Mathematics*, 2022, Vol. 10, No. 4, Art. No. 658, 11 pp.

[10] M. Manev, Yamabe solitons on conformal almost contact complex Riemannian manifolds with vertical torse-forming vector field, *Axioms*, 2023, Vol. 12, No. 1, Art. No. 44, 10 pp.

[11] M. Manev, Pairs of associated Yamabe almost solitons with vertical potential on almost contact complex Riemannian manifolds, *Mathematics*, 2023, Vol. 11, No. 13, Art. No. 2870, 12 pp.

[12] B. Chow, P. Lu, L. Ni, Hamilton's Ricci Flow, In *Graduate Studies in Mathematics*, Vol. 77; American Mathematical Society: Providence, RI; Science Press: Beijing, New York, 2006.

[13] G. Ganchev, V. Mihova, K. Gribachev, Almost contact manifolds with B-metric, *Math. Balkanica (N.S.)*, 1993, Vol. 7, 261–276.

[14] M. Manev, Contactly conformal transformations of general type of almost contact manifolds with B-metric. Applications, *Math. Balkanica (N.S.)*, 1997, Vol. 11, 347–357.

[15] S. Ivanov, H. Manev, M. Manev, Sasaki-like almost contact complex

Riemannian manifolds, *J. Geom. Phys.*, 2016, Vol. 105, 136–148.

[16] K. Yano, On torse forming direction in a Riemannian space, *Proc. Imp. Acad. Tokyo*, 1944, Vol. 20, 340–345.

[17] A. Mihai, I. Mihai, Torse forming vector fields and exterior concurrent vector fields on Riemannian manifolds and applications, *J. Geom. Phys.*, 2013, Vol. 73, 200–208.

[18] J. Schouten, *Ricci Calculus. An Introduction to Tensor Analysis and Its Geometrical Applications.* Springer-Verlag, Berlin, Heidelberg, 1954.

Mancho Manev [1,2]
[1] Paisii Hilendarski University of Plovdiv,
Faculty of Mathematics and Informatics,
236 Bulgaria Blvd., 4003 Plovdiv, Bulgaria
[2] Medical University – Plovdiv, Faculty of Pharmacy,
15A Vasil Aprilov Blvd., 4002 Plovdiv, Bulgaria
Corresponding author: `mmanev@uni-plovdiv.bg`