EXAMPLES OF EVENTS THAT ARE PAIRWISE INDEPENDENT BUT NOT MUTUALLY INDEPENDENT

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Abstract. *The concept of independence is central to Probability Theory. When we have more than 2 random events we have different types of independence: ensemble independence and pairwise independence. In this work, we will give examples of N random events* $A_1, A_2, A_3, \ldots, A_N$, $N \geq 3$, *which are not independent in aggregate, but for each of their subset of* k, $1 < k < N$ *events* $A_{n_1}, A_{n_2}, A_{n_3}, \ldots, A_{n_k}$ *these events are mutually independent (and therefore pairwise independent). The examples are constructed using a symmetric die with* 2^{N-1} *faces, which are colored N different colors, conventionally denoted by the numbers* $1, 2, 3, \ldots, N$.

Key words: Independence, Pairwise independence, Mutual independence.

We will describe the scheme for $N = 3$ colors, $N = 4$ colors (to better illustrate the idea) and for any number of $N \geq 3$ colors.

We will use the well-known combinatorial equality

$$\sum_{k=-\infty}^{+\infty} C_n^{2k} = \sum_{k=-\infty}^{+\infty} C_n^{2k+1} = 2^{n-1}$$

(for $k < 0$ or $2k > n$ $C_n^{2k} = C_n^{2k+1} = 0$ so the sums are actually finite).

First we consider experiment with odd number of colors placed on each side of the dice: on $N = C_N^1$ walls there is 1 color applied, on $\frac{N.(N-1).(N-2)}{6} = C_N^3$ walls have 3 colors applied, on $\frac{N.(N-1).(N-2).(N-3).(N-4)}{120} = C_N^5$ walls of dice have 5 colors applied, and so on.

1) The dice is a regular tetrahedron. $N = 3$ colors. $4 = \sum_{k=-\infty}^{+\infty} C_3^{2k+1} = 2^{3-1}$ (Figure 1, Bernstein Example).

On the sides of the dice there are 3 colors – white (1), green (2) and red (3). There is one color on $3 = C_3^1$ walls and 3 colors on $1 = C_3^3$ walls.

Then

$$|A_1| = |A_2| = |A_3| = 2 \longrightarrow P(A_1) = P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$$
$$|A_1A_2| = |A_1A_3| = |A_2A_3| = 1 \longrightarrow$$
$$P(A_1A_2) = \frac{1}{4} = P(A_1).P(A_2),$$
$$P(A_1A_3) = \frac{1}{4} = P(A_1).P(A_3),$$
$$P(A_2A_3) = \frac{1}{4} = P(A_2).P(A_3)$$
$$|A_1A_2A_3| = 1 \longrightarrow P(A_1A_2A_3) = \frac{1}{4} \quad \neq \quad P(A_1).P(A_2).P(A_3) = \frac{1}{8}$$

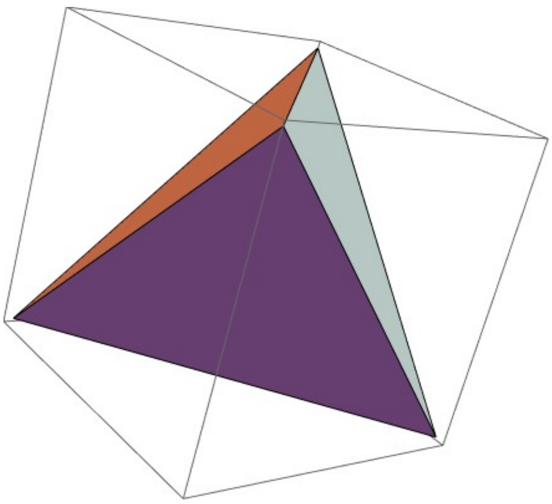

Figure 1.

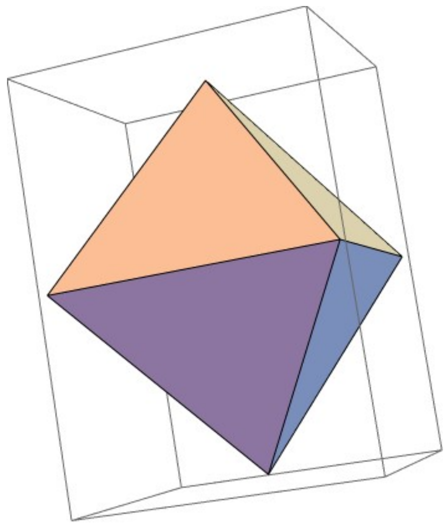

Figure 2.

2) The dice is a regular octahedron. $N = 4$ colors. $8 = \sum_{k=-\infty}^{+\infty} C_4^{2k+1} = 2^{4-1}$ (Figure 2).

There are 4 colors on the sides of the dice – white (1), green (2), red (3) and blue (4). There is one color on $4 = C_4^1$ walls and 3 colors on $4 = C_4^3$ walls.

Then

$$|A_1| = |A_2| = |A_3| = |A_4| = 4 \longrightarrow P(A_1) = P(A_2) = P(A_3) = \frac{4}{8} = \frac{1}{2}$$
$$|A_1A_2| = |A_1A_3| = |A_1A_4| = |A_2A_3| = |A_2A_4| = |A_3A_4| = 2 \longrightarrow$$

$$P(A_1A_2) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_2),$$
$$P(A_1A_3) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_3),$$
$$P(A_1A_4) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_4),$$
$$P(A_2A_3) = \frac{2}{8} = \frac{1}{4} = P(A_2).P(A_3),$$
$$P(A_2A_4) = \frac{2}{8} = \frac{1}{4} = P(A_2).P(A_4),$$
$$P(A_3A_4) = \frac{2}{8} = \frac{1}{4} = P(A_3).P(A_4).$$
$$|A_1A_2A_3| = |A_1A_2A_4| = |A_1A_3A_4| = |A_2A_3A_4| = 1 \longrightarrow$$
$$P(A_1A_2A_3) = \frac{1}{8} = P(A_1).P(A_2).P(A_3),$$
$$P(A_1A_2A_4) = \frac{1}{8} = P(A_1).P(A_2).P(A_4),$$
$$P(A_1A_3A_4) = \frac{1}{8} = P(A_1).P(A_3).P(A_4),$$
$$P(A_2A_3A_4) = \frac{1}{8} = P(A_2).P(A_3).P(A_4),$$
$$P(A_1A_2A_3A_4) = 0 \neq P(A_1).P(A_2).P(A_3).P(A_4) = \frac{1}{16}$$

3) General case. N colors. $N \geq 3$.

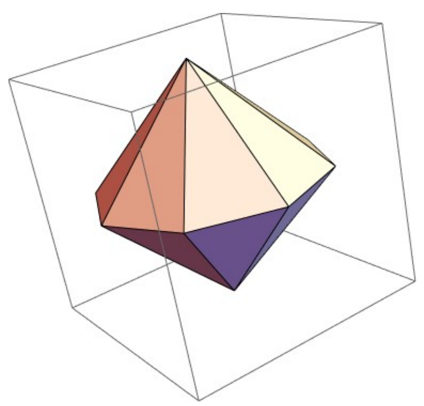

Figure 3. Bipyramid with $16 = 2^{5-1}$ walls. Its walls will be colored with $N = 5$ colors.

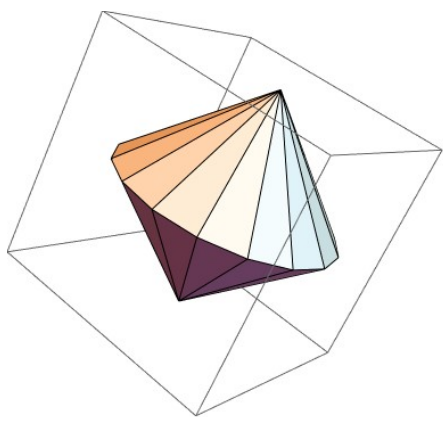

Figure 4. Bipyramid with $32 = 2^{6-1}$ walls. Its walls will be colored with $N = 6$ colors.

The dice is a regular bipyramid. The base of a bipyramid is a regular polygon with 2^{N-2} sides, and a bipyramid has $\sum\limits_{k=-\infty}^{+\infty} C_N^{2k+1} = 2^{N-1}$ faces that are equal isosceles triangles.

The sides of the dice have N colors.

On $N = C_N^1$ walls there is one color, on $\frac{N.(N-1).(N-2)}{6} = C_N^3$ walls there are 3 colors, on $\frac{N.(N-1).(N-2).(N-3).(N-4)}{120} = C_N^5$ walls there are 5 colors, and so on.

The total number of colored walls is $C_N^1 + C_N^3 + C_N^5 + \cdots = 2^{N-1}$. Let $A_k = \{\text{the } k^{\text{th}} \text{ color has appeared on the wall}\}$, $k = 1, 2, 3, 4, \ldots, N$.

$$|A_k| = C_{N-1}^0 + C_{N-1}^2 + C_{N-1}^4 + C_{N-1}^6 + \cdots = 2^{N-2}$$
$$\longrightarrow P(A_k) = \frac{2^{N-2}}{2^{N-1}} = \frac{1}{2}$$
$$|A_{k_1}.A_{k_2}| = C_{N-2}^{-1} + C_{N-2}^1 + C_{N-2}^3 + C_{N-2}^5 + \ldots = 2^{N-3}$$
$$\longrightarrow P(A_{k_1}.A_{k_2}) = \frac{2^{N-3}}{2^{N-1}} = \frac{1}{4} = P(A_{k_1}).P(A_{k_2})$$
$$|A_{k_1}.A_{k_2}.A_{k_3}| = C_{N-3}^{-2} + C_{N-3}^0 + C_{N-3}^2 + C_{N-3}^4 + \cdots = 2^{N-4}$$
$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3}) = \frac{2^{N-4}}{2^{N-1}} = \frac{1}{8} = P(A_{k_1}).P(A_{k_2}).P(A_{k_3})$$

(The first combinations are for the one-color walls, the second are for the three-color walls, the third are for the five-color walls, etc.)

$$\ldots$$
$$|A_{k_1}.A_{k_2}.A_{k_3} \ldots A_{k_s}| = C_{N-s}^{1-s} + C_{N-s}^{3-s} + C_{N-s}^{5-s} + C_{N-s}^{7-s} + \cdots = 2^{N-s-1}$$
$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3}.\ \ldots\ A_{k_s}) = \frac{2^{N-s-1}}{2^{N-1}}$$
$$= \frac{1}{2^s} = P(A_{k_1}).P(A_{k_2}).P(A_{k_3}) \ldots P(A_s)$$
$$\ldots$$
$$|A_{k_1}.A_{k_2}.A_{k_3} \ldots A_{k_{N-1}}| = 1$$
$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3} \ldots A_{k_{N-1}}) = \frac{1}{2^{N-1}}$$
$$= P(A_{k_1}).P(A_{k_2}).P(A_{k_3}) \ldots P(A_{N-1})$$
$$|A_1.A_2.A_3 \ldots A_N| = \begin{cases} 0, & N \text{ even number} \\ 1 & \text{odd number} \end{cases}$$

$$\longrightarrow P(A_1.A_2.A_3\ldots A_N) = \begin{cases} 0, & N \text{ even number} \\ \dfrac{1}{2^{N-1}} & N \text{ odd number} \end{cases}$$

$$\longrightarrow P(A_1.A_2.A_3\ldots A_N) \neq P(A_1).P(A_2).P(A_3)\ldots.P(A_N) = \frac{1}{2^N}$$

Now let's consider a similar experiment, but an even number of colors will be placed on each side of the dice: on $1 = C_N^0$ walls there are 0 colors applied (the wall remains unpainted – for example black), on $\frac{N.(N-1)}{2} = C_N^2$ walls have 2 colors applied, on $\frac{N.(N-1).(N-2).(N-3)}{24} = C_N^4$ walls have 4 colors applied, and so on.

Again, we consider the tetrahedron, octahedron, and general case:

4) The dice is a regular tetrahedron. $N = 3$ colors. $4 = \sum\limits_{k=-\infty}^{+\infty} C_3^{2k} = 2^{3-1}$. 3 colors are applied to the sides of the dice – white (1), green (2) and red (3). On $1 = C_3^0$ walls there are zero colors, on $3 = C_3^2$ walls there are 2 colors applied. Then

$$\begin{aligned}
&|A_1| = |A_2| = |A_3| = 2 \longrightarrow P(A_1) = P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2} \\
&|A_1A_2| = |A_1A_3| = |A_2A_3| = 1 \longrightarrow \\
&P(A_1A_2) = \frac{1}{4} = P(A_1).P(A_2), \\
&P(A_1A_3) = \frac{1}{4} = P(A_1).P(A_3), \\
&P(A_2A_3) = \frac{1}{4} = P(A_2).P(A_3), \\
&|A_1A_2A_3| = 0 \longrightarrow P(A_1A_2A_3) = 0 \neq P(A_1).P(A_2).P(A_3) = \frac{1}{8}
\end{aligned}$$

5) The dice is a regular octahedron. $N = 4$ colors. $8 = \sum\limits_{k=-\infty}^{+\infty} C_4^{2k} = 2^{4-1}$.

There are 4 colors on the sides of the dice – white (1), green (2), red (3) and blue (4). There are zero colors on the $1 = C_4^0$ faces, 2 colors on the $6 = C_4^2$ faces, and four colors on the $1 = C_4^4$ faces.

Then

$$|A_1| = |A_2| = |A_3| = |A_4| = 4 \longrightarrow P(A_1) = P(A_2) = P(A_3) = \frac{4}{8} = \frac{1}{2}$$
$$|A_1A_2| = |A_1A_3| = |A_1A_4| = |A_2A_3| = |A_2A_4| = |A_3A_4| = 2 \longrightarrow$$
$$P(A_1A_2) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_2),$$
$$P(A_1A_3) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_3),$$
$$P(A_1A_4) = \frac{2}{8} = \frac{1}{4} = P(A_1).P(A_4),$$
$$P(A_2A_3) = \frac{2}{8} = \frac{1}{4} = P(A_2).P(A_3),$$
$$P(A_2A_4) = \frac{2}{8} = \frac{1}{4} = P(A_2).P(A_4),$$
$$P(A_3A_4) = \frac{2}{8} = \frac{1}{4} = P(A_3).P(A_4),$$
$$|A_1A_2A_3| = |A_1A_2A_4| = |A_1A_3A_4| = |A_2A_3A_4| = 1 \longrightarrow$$
$$P(A_1A_2A_3) = \frac{1}{8} = P(A_1).P(A_2).P(A_3),$$
$$P(A_1A_2A_4) = \frac{1}{8} = P(A_1).P(A_2).P(A_4),$$
$$P(A_1A_3A_4) = \frac{1}{8} = P(A_1).P(A_3).P(A_4),$$
$$P(A_2A_3A_4) = \frac{1}{8} = P(A_2).P(A_3).P(A_4),$$
$$P(A_1A_2A_3A_4) = \frac{1}{8} \neq P(A_1).P(A_2).P(A_3).P(A_4) = \frac{1}{16}$$

6) General case. N colors. $N \geq 3$.

The dice is a regular bipyramid. The base of a bipyramid is a regular polygon with 2^{N-2} sides, and a bipyramid has $\sum\limits_{k=-\infty}^{+\infty} C_N^{2k} = 2^{N-1}$ faces that are equal isosceles triangles.

The sides of the dice have N colors. On $1 = C_N^0$ walls there is one color, on $\frac{N.(N-1)}{2} = C_N^2$ sides there are 2 colors, on $\frac{N.(N-1).(N-2).(N-3)}{24} = C_N^4$ sides there are 4 colors, and so on.

The total number of colored walls is $C_N^0 + C_N^2 + C_N^4 + \cdots = 2^{N-1}$.

Let $A_k = \{\text{the } k^{\text{th}} \text{ color has appeared on the side}\}$, $k = 1, 2, 3, 4, \ldots, N$.

$$|A_k| = C_{N-1}^{-1} + C_{N-1}^{1} + C_{N-1}^{3} + C_{N-1}^{5} + \cdots = 2^{N-2}$$

$$\longrightarrow P(A_k) = \frac{2^{N-2}}{2^{N-1}} = \frac{1}{2}$$

$$|A_{k_1}.A_{k_2}| = C_{N-2}^{-2} + C_{N-2}^{0} + C_{N-2}^{2} + C_{N-2}^{4} + \cdots = 2^{N-3}$$

$$\longrightarrow P(A_{k_1}.A_{k_2}) = \frac{2^{N-3}}{2^{N-1}} = \frac{1}{4} = P(A_{k_1}).P(A_{k_2})$$

$$|A_{k_1}.A_{k_2}.A_{k_3}| = C_{N-3}^{-3} + C_{N-3}^{-1} + C_{N-3}^{1} + C_{N-3}^{3} + \cdots = 2^{N-4}$$

$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3}) = \frac{2^{N-4}}{2^{N-1}} = \frac{1}{8} = P(A_{k_1}).P(A_{k_2}).P(A_{k_3})$$

(the first combinations are for the single-color walls, the second are for the three-color walls, the third are for the five-color walls, etc.)

$$\ldots$$

$$|A_{k_1}.A_{k_2}.A_{k_3}\ldots A_{k_s}| = C_{N-s}^{-s} + C_{N-s}^{2-s} + C_{N-s}^{4-s} + C_{N-s}^{6-s} + \cdots = 2^{N-s-1}$$

$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3}.\ldots.A_{k_s}) = \frac{2^{N-s-1}}{2^{N-1}}$$

$$= \frac{1}{2^s} = P(A_{k_1}).P(A_{k_2}).P(A_{k_3}).\ldots.P(A_s)$$

$$\ldots$$

$$|A_{k_1}.A_{k_2}.A_{k_3}.\ldots.A_{k_{N-1}}| = 1$$

$$\longrightarrow P(A_{k_1}.A_{k_2}.A_{k_3}\ldots A_{k_{N-1}}) = \frac{1}{2^{N-1}}$$

$$= P(A_{k_1}).P(A_{k_2}).P(A_{k_3}).\ldots.P(A_{N-1})$$

$$|A_1.A_2.A_3.\ldots.A_N| = \begin{cases} 1, & N \text{ even} \\ 0, & N \text{ odd} \end{cases}$$

$$\longrightarrow P(A_1.A_2.A_3.\ldots.A_N) = \begin{cases} \frac{1}{2^{N-1}}, & N \text{ even} \\ 0, & N \text{ odd} \end{cases}$$

$$\longrightarrow P(A_1.A_2.A_3.\ldots.A_N) \neq P(A_1).P(A_2).P(A_3).\ldots.P(A_N) = \frac{1}{2^N}$$

And in this case we got events that are not independent in aggregate, but for each of their subset of k, $k < N$ events $A_{k_1}.A_{k_2}.A_{k_3}.\ \ldots\ .A_{k_{N-1}}$ these events are collectively independent. The difference is the probability $P(A_1.A_2.A_3.\ \ldots\ .A_N)$.

To illustrate better the complexity of the i situation with independence in these trials, we will consider a combination of the two trials (with an even and an odd number of wall colors) by combining them into one:

Consider the following experiment: we have two symmetric dice with 2^{N-1} faces each, which are colored N different colors, the first being colored according to the "odd rule" (with 1, 3, 5, … colors on the faces) and the second is colored according to the "even rule" (with 0, 2, 4, … colors on the walls) as described above. We toss a fair coin and if it lands heads (event H_1 with probability $\frac{1}{2}$) then we roll the first die with the odd number of colors on the walls, and if it lands heads (event H_2 with probability $\frac{1}{2}$) then we roll the second die with the even number of colors on the walls. Then for each of the events $A_{k_1}.A_{k_2}.A_{k_3}.\ \dots\ .A_{k_s}$ we have (by the formula for the total probability)

$$\begin{aligned} P(A_{k_1}.A_{k_2}.A_{k_3}.\ \dots\ .A_{k_s}) = &\ P(H_1).P(A_{k_1}.A_{k_2}.A_{k_3}.\ \dots\ .A_{k_s}|H_1)+ \\ &+ P(H_2).P(A_{k_1}.A_{k_2}.A_{k_3}.\ \dots\ .A_{k_s}|H_2) \\ = &\ \frac{1}{2^s}, \quad s = 1,\, 2,\, 3,\, 4,\, \dots,\, N. \end{aligned}$$

The case $s = N$ differs from the others, but this probability is trivially computed here as well.

We obtained that the events A_1, A_2, A_3, …, A_N are collectively independent in this trial! In the two separate experiments (rolling a die with an even or odd number of wall colors) we have no independence in aggregate, but when they are combined the dependence disappears!

In fact, this trial is equivalent to the following trial with only one symmetric die: given a symmetric die with 2^N faces that are colored N different colors, with $1 = C_N^0$ faces having zero colors, $N = C_N^1$ faces there is 1 color applied, on $\frac{N.(N-1)}{2} = C_N^2$ walls there are 2 colors applied, on $\frac{N.(N-1).(N-2)}{6} = C_N^3$ there are 3 colors applied, 4 colors are applied to $\frac{N.(N-1).(N-2).(N-3)}{24} = C_N^4$ walls, etc. (we have 2^{N-1} walls each colored with an even number of colors and 2^{N-1} walls each colored with an odd number of colors).

The dice is thrown and it is counted on which wall it landed. If the die lands on a side with an odd number of colors, we have an event H1 with probability $\frac{1}{2}$. If the die falls on a side with an even number of colors, we have an event H2 with probability $\frac{1}{2}$. The probabilities of the remaining events are calculated according to the above scheme and turn out to be again $P(A_{k_1}.A_{k_2}.A_{k_3}.\ \ldots\ .A_{k_s}) = \frac{1}{2^s}$, $s = 1, 2, 3, 4, \ldots, N$.

Comments:

Instead of a symmetric dice to generate the trials, another physical object generating the 2^N outcomes with equal probabilities (e.g. a roulette wheel) can be used. Other combinatorial dependencies can be used to give other schemes illustrating the complexity of the notions of "independence" and "mutual independence".

Examples of real dice (for illustration):

Figure 5.

Figure 6.

Figure 7. Examples of symmetric dice.
Some are Platonic solids, others are bipyramidal in shape.

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