USING COMPUTER ALGEBRA SYSTEMS TO COMPUTE LIMITS OF FUNCTIONS BY THE L'HOSPITAL'S THEOREMS

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Abstract. *Computing limits on functions by applying L'Hospital's rules to some problems can be a difficult and time-consuming process. In this work, we present procedures for checking indeterminate of the form $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$, as well as procedures for iteratively and recursively applying L'Hospital's rules. This, in turn, helps the work of the students in the Mathematical Analysis classes as they can thus concentrate not on the algebraic calculations but on the idea of the theory under consideration.*

1. Introduction

Mathematical analysis is a basic discipline that is studied by students of informatics at the Faculty of Mathematics and Informatics of Paisii Hilendarski University of Plovdiv. Along with the study of basic concepts such as limit, derivative, continuity, integral, etc., the course of mathematical analysis also studies the theorems of L'Hospital, as an immediate application of the concept of derivative, which enable us to calculate limits of functions when we have indeterminate of the type $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$. It is well known that some problems related to finding the limit of a function applying L'Hospital's theorems require rather long computations that take considerable time. In order to increase students interest in the material, in addition to finding function bounds by "hand" computation, we demonstrate procedures using the computer algebra system Maple to check indeterminate of the form $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$. Using appropriate examples, we demonstrate their iterative and recursive application of L'Hospital's theorems, as well as the time required for the computer to perform the necessary predictions for each species.

2. Finding the limit of a function by the L'Hospital's theorems

Let us consider a problem of finding the limit of a function of a one variable.

Example 2.1. *Find the limit* $\lim\limits_{x \to 0} \dfrac{1 - \cos(x^2)}{\ln(1 - x^4)}$.

Solution: Immediately we see that if we replace x by 0 in the given limit we have an indeterminate of the form $\left[\frac{0}{0}\right]$. We can apply L'Hospital's theorems. For this purpose, we find the derivative of the numerator and the derivative of the denominator and the given limit we can then the given limit has the form:
$$\lim_{x \to 0} \frac{-2(1 - x^4)\sin(x^2)}{x^2}.$$

Using $\lim\limits_{x \to 0} \dfrac{\sin\left(x^2\right)}{x^2} = 1$ we get that $\lim\limits_{x \to 0} \dfrac{-2(1-x^4)\sin(x^2)}{x^2} = -\dfrac{1}{2}$.

As we well know, L'Hospital's theorems can only be applied directly to limits where we have the above two types of basic indeterminacies. Otherwise, if we have indeterminacies of the form $[\infty - \infty]$, $[0.\infty]$, $[1^\infty]$ etc. it is necessary to represent the function whose limit we seek in a suitable way so as to obtain either of the two basic types of uncertainties. We define two procedures in Maple that check if there are indeterminates of the form either $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$

```
ULZ := proc(a, f, g)
local u;
if (limit(f(x), x = a) = 0) and limit(g(x), x = a) = 0)
then true
else false
end if
end proc;
ULI := proc(a, f, g)
local u;
if (limit(f(x), x = a) = infinity and limit(g(x), x = a) = infinity)
then true
else false
end if
end proc
```

We check what type of indeterminates we have

$$\mathrm{ULZ}(0, f, g); \mathrm{ULZ}(1, f, g); \mathrm{UZI}(0, f, g);$$

Maple returns

true, false, false

From Maple's last answer we see that in the example under consideration we have an indeterminate of the form $\left[\frac{0}{0}\right]$ at 0.

With the next procedure, we will iteratively apply the theorems of L'Hospital to compute the limit under consideration. We introduce the following procedure in Maple:

$\mathrm{LTI} := \mathrm{proc}(a, f, g)$
$\mathrm{locali}, h1, h2, p;$
for i from 0 while $\mathrm{ULZ}(a, f, g)$ or $\mathrm{ULI}(a, f, g)$ do
$h1 := (D^{(i)})(f); h2 := (D^{(i)})(g); \mathrm{print}\left(i, \frac{h1(x)}{h2(x)}\right);$
if $(\mathrm{not}((\mathrm{ULI}(a, (D^{(i)})(f), (D^{(i)})(g)))$ and $\mathrm{not}(\mathrm{ULZ}(a, (D^{(i)})(f), (D^{(i)})(g)))$
then $h1 := (D^{(i)})(f); h2 := (D^{(i)})(g) : p := i : \mathrm{break}$
end if :
end do :
$\mathrm{limit}\left(\frac{h1(x)}{h2(x)}, x = a\right) :$
end proc;

We compute the limit using the iteratively defined procedure *LTI*

$$\mathrm{LTI}(0, f, g)$$

We get the following answer from Meple:

$$0, \qquad \frac{1 - \cos(x^2)}{\ln(-x^3 + 1)}$$

$$1, \qquad \frac{-\sin(x^2)(-x^4 + 1)}{2x^2}$$

$$2, \qquad \frac{4\cos(x^2)x^2 + 2\sin(x^2)}{-\frac{12x^2}{1-x^4} - \frac{16x^6}{(1-x^4)^2}}$$

$$3, \qquad \frac{-8\sin(x^2)x^3 + 12\cos(x^2)x}{-\frac{24}{1-x^4} - \frac{144x^5}{(1-x^4)^2} - \frac{128x^9}{(1-x^4)^3}}$$

$$4, \qquad \frac{-16\cos(x^2)x^4 - 48\sin(x^2)x^2 + 12\cos(x^2)}{-\frac{24}{1-x^4} - \frac{816x^4}{(1-x^4)^2} - \frac{2304x^8}{(1-x^4)^3} - \frac{1536x^8}{(1-x^4)^4}}$$

$$-\frac{1}{2}$$

Let us note that Maple does not recognize in the procedure defined by us the limit $\lim\limits_{x\to 0} \frac{\sin(x^2)}{x^2} = 1$. The software continues to calculate the derivative till it reaches a determined fraction and presents the answer.

With the following example we will illustrate the recursive application of L'Hospital theorems and compare the speed of the two methods (iterative and recursive)

Example 2.2. *Find the limit* $\lim\limits_{x\to 0} \frac{(1-\cos(x^2))\operatorname{arctg}(x)}{\sin(x^4)\ln(1-x)}$.

We define the functions $F : x \to (1 - \cos(x^2))\arctan(x)$ and $G : x \to \sin(x^4)\ln(1-x)$. Starting the procedure $LTI(0, F, G)$ we get

$$0, \qquad \frac{1-\cos(x^2)\arctan(x)}{\sin(x^4)\ln(1-x)}$$

$$1, \qquad \frac{2\sin(x^2)x\arctan(x) + \frac{1-\cos(x^2)}{1+x^2}}{4\cos(x^3)x^3\ln(1-x) - \frac{\sin(x^4)}{1-x}}$$

$$2, \qquad \dots\dots\dots\dots\dots\dots\dots$$

$$.. \qquad \dots\dots\dots\dots\dots\dots\dots$$

$$5, \qquad \dots\dots\dots\dots\dots\dots\dots$$

$$-\frac{1}{2},$$

where the derivatives $F^{(n)}(x)$ and $G^{(n)}(x)$ for $n = 2, 3, 4, 5$ are too complicated.

We now write in Maple recursive procedure to apply L'Hospital theorems.

```
LTR := proc(a, f, g, i)
local h1, h2; print (i, f(x)/g(x)) :
if (ULZ(a, f, g) or ULI(a, f, g))
then thisproc(a, D^(1)(f), D^(1)(g), i + 1) :
else limit (f(x)/g(x), x = a) :  end if
end proc;
```

We see that the recursive procedure is written with fewer commands.

We will compare the time needed to calculate a limit with the help of the two procedures, the iterative one LTI and the recursive one LTR. We take the processor time $st := time()$ before the start of the procedure

and after it finished we calculate, how much time it has taken.

$$\text{st} := \text{time}(); \text{LTI}(0, \text{F}, \text{G}); \text{time}() - \text{st};$$
$$\text{st} := \text{time}(); \text{LTR}(0, \text{F}, \text{G}, 0); \text{time}() - \text{st};$$

The software computes the limit presenting one and same sequences of successive derivatives and returns that the computer time needed to calculate iteratively is 0.109 and recursively is 0.031. This simple example with a few derivatives, needed to find the answer, shows that the recursive procedure works faster.

Example 2.3. *Find the limit* $\lim\limits_{x \to 0} \dfrac{\sin(x) - \sum\limits_{k=1}^{40} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}}{e^x - \sum\limits_{k=0}^{80} \frac{x^k}{k!}}.$

We define the functions $\varphi : x \to \sin(x) - \sum\limits_{k=1}^{40} \dfrac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$ and $\psi : x \to e^x - \sum\limits_{k=0}^{80} \dfrac{x^k}{k!}.$ Starting the procedures

$$\text{st} := \text{time}(); \text{LTI}(0, \varphi, \psi); \text{time}() - \text{st};$$
$$\text{st} := \text{time}(); \text{LTR}(0, \varphi, \psi, 0); \text{time}() - \text{st};$$

After calculating 81 derivatives we get that the limit is equal to 1 and it takes 41.344 seconds by the the iterative procedure and 0.562 seconds by the recursive one.

Let us pay attention that the calculation of the time, required to perform the calculations, depends on both the computer and the time when the calculations was performed. It is possible that if at the moment of calculation the computer is performing various processes in the background, then we will get a different value for the elapsed time.

3. Conclusion

With what has been illustrated so far, we can conclude that math software, although they perform complex and time-consuming calculations accurately and quickly, have their limitations. In example 1, we saw that the software failed to realize that it is not necessary to look for derivatives of higher order than 1. The same is true when a product of functions is obtained, so that some of them do not give indeterminacy of species.

On the other hand, the use of software contributes to the acquisition of basic programming skills, creates additional interest among students in the topics under consideration, and provides them with an opportunity to easily check the calculations they have made by hand.

Whenever possible and computer memory allows it is good to use recursive procedures instead of iterative ones.

Acknowledgments

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