NUMERICAL SOLVING OF THE SINE-GORDON EQUATION

Pavlina Atanasova, Valentin Georgiev

Abstract. *The sine-Gordon equation is a nonlinear partial differential equation, describing a multitude of physical phenomena. In this paper, the (1+1)-dimensional sine-Gordon equation is numerically solved, utilizing the Crank-Nicolson method and tridiagonal sweep. The results are verified by comparison with analytical solutions. An error estimate is presented.*

Key words: Partial differential equation, Nonlinearity, Sine-Gordon equation, Crank-Nicolson.

Mathematics Subject Classification: 65M06, 65N06.

1. Introduction

The sine-Gordon equation

$$u_{xx} - u_{tt} = \sin(u) \tag{1}$$

is an important nonlinear partial differential equation, first appearing in differential geometry [1]. Its name comes from its linearization, known in quantum mechanics as the Klein-Gordon equation $u_{xx} - u_{tt} = u$.

Among its many applications, some are the description of self-induced transparency [2], the dynamics in long Josephson junctions [3], the description of magnetic domain wall dynamics [4], the classical model of one-dimensional dislocation theory [1].

In [5] a classification of some analytical solutions is presented. The main feature of most solutions of (1) is the form of the partial derivative with respect to x, it being a soliton – a localized disturbance or pulse that retains its shape after interacting with other solitons [1].

A couple of the solutions have been used as a source of initial and boundary conditions:

1. Travelling wave solution

$$u(x,t) = \pm 4 \arctan\left[c \exp\left(\frac{x \pm vt}{\sqrt{1-v^2}}\right)\right], \tag{2}$$

where $v = \frac{\sqrt{s^2-1}}{s}$, c and $s^2 > 1$ are constants and all sign combinations are possible.

2. Two-soliton solution

$$u(x,t) = \pm 4 \arctan\left[\frac{\sqrt{s^2-1}}{s} \frac{\sinh(sx + c_1)}{\cosh(\sqrt{s^2-1} \cdot t + c_2)}\right], \tag{3}$$

where $s^2 > 1$ and $c_1,\ c_2$ are constants of integration.

3. Soliton-Antisoliton solution

$$u(x,t) = -4 \arctan\left[\frac{s}{\sqrt{s^2-1}} \frac{\sinh(\sqrt{s^2-1} \cdot t + c_2)}{\cosh(sx + c_1)}\right], \tag{4}$$

where $s^2 > 1$ and $c_1,\ c_2$ are constants of integration.

4. Breather solution

$$u(x,t) = -4 \arctan\left[\frac{s}{\sqrt{1-s^2}} \frac{\sin(\sqrt{1-s^2} \cdot t + c_2)}{\cosh(sx + c_1)}\right], \tag{5}$$

where $s^2 < 1$ and $c_1,\ c_2$ are constants of integration.

In this paper we have considered equation (1) with the initial conditions

$$u(x,0) = \varphi(x),\ x \in [a,b]$$
$$u_t(x,0) = \psi(x),\ x \in [a,b]$$

and the boundary conditions

$$u(a,t) = \mu_L(t),\ t \in [0,T]$$
$$u(b,t) = \mu_R(t),\ t \in [0,T].$$

2. Numerical Approach

The method used in this paper is a variation on the Crank-Nicolson method [6, 7], which is a second order accuracy finite difference method. It can be shown that in diffusion equations (and many others) it is unconditionally stable [8]. It is classically used in problems involving u_t as the highest derivative of u in terms of t. Let us instead consider the equation

$$u_{tt} = F\left(u, x, t, u_x, u_{xx}\right)$$

on the intervals $x \in [a, b]$, $t \in [0, T]$ and let us create a uniform mesh with steps $h = \frac{b-a}{n}$ for x and $\tau = \frac{T}{m}$ for t, $m, n \in \mathbb{N}$. Using the notation $x_i = a + ih, t_j = j\tau$, $u(x_i, t_j) = u_i^j$ and $F_i^j = F$ evaluated for x_i, t_j and u_i^j $(i = \overline{0, n}, j = \overline{0, m})$ we get

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2} = \frac{1}{2}\left[F_i^{j+1} + F_i^{j-1}\right].$$

This is an implicit method – it leads to a system of algebraic equations for the layer $t = (j+1)\tau$, which in general is nonlinear. If F is linear, then the system will be tridiagonal, allowing to be solved efficiently by using a tridiagonal sweep [9].

Our approach of providing the scheme is to separate $F = u_{xx} - \sin(u)$ into a linear (u_{xx}) and a nonlinear part $(-\sin(u))$ and employ linear interpolation on u_{xx}. If $L_2(x_i, t)$ denotes the Lagrange interpolation polynomial of u_{xx} on the interval $t \in [t_{j-1}, t_{j+1}]$ with nodes $(x_i, t_{j-1}, {u_{xx}}_i^{j-1})$ and $(x_i, t_{j+1}, {u_{xx}}_i^{j+1})$ we get the following estimate

$$|u_{xx} - L_2(x_i, t)| \leq \frac{M_3}{3!}|(t - t_{j-1})(t - t_{j+1})| = O(\tau^2)$$

where $M_3 = \max\limits_{t \in [t_{j-1}, t_{j+1}]} \{u_{xxttt}|_{x=x_i, t}\}$. The advantage to this approach is removing the necessity of solving a nonlinear system of equations, while keeping the implicit nature of the method.

Let us consider the discretization of equation (1)

$$ {u_{tt}}_i^j = {u_{xx}}_i^j - \sin\left(u_i^j\right) \tag{6}$$

and then, applying midpoint second order accuracy finite differences and the linear interpolation of u_{xx} at the point (x_i, t_j), the left hand side (LHS) and right hand side (RHS) become

$$LHS = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2}$$

$$RHS = \frac{1}{2}\left[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{h^2}\right] - \sin\left(u_i^j\right).$$

After some algebraic transformations we are left with

$$\begin{aligned} &\frac{\tau^2}{2h^2} u_{i-1}^{j+1} - \left(1 + \frac{\tau^2}{h^2}\right) u_i^{j+1} + \frac{\tau^2}{2h^2} u_{i+1}^{j+1} = \\ &\qquad = u_i^{j-1} - 2u_i^j - \frac{\tau^2}{2h^2}(u_{i-1}^{j-1} - 2u_i^{j-1} + u_{i+1}^{j-1}) + \tau^2 \sin\left(u_i^j\right) \end{aligned} \tag{7}$$

for $i = \overline{1, n-1}, j = \overline{1, m-1}$. The stencil is shown on Figure 1.

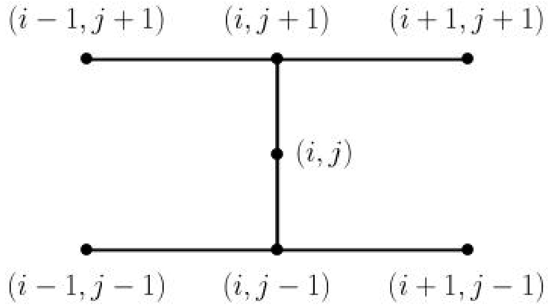

Figure 1. Stencil for the Crank-Nicolson method

For simplicity, let us denote the RHS of (7) as z_i^{j+1}. Then we can express the scheme as a system of linear algebraic equations

$$\boldsymbol{A}\boldsymbol{u}^{j+1} = \boldsymbol{z}^{j+1},$$

where

$$\boldsymbol{A} = \begin{pmatrix} -\left(1+\frac{\tau^2}{h^2}\right) & \frac{\tau}{2h^2} & 0 & \dots & & 0 & \\ \frac{\tau^2}{2h^2} & -\left(1+\frac{\tau^2}{h^2}\right) & \frac{\tau}{2h^2} & & & & \\ & \vdots & & \ddots & & \vdots & \\ & 0 & & \dots & \frac{\tau^2}{2h^2} & -\left(1+\frac{\tau^2}{h^2}\right) & \frac{\tau}{2h^2} \\ & & & & 0 & \frac{\tau^2}{2h^2} & -\left(1+\frac{\tau^2}{h^2}\right) \end{pmatrix},$$

$$\boldsymbol{u}^{j+1} = (u_1^{j+1}, u_2^{j+1}, \dots u_{n-2}^{j+1}, u_{n-1}^{j+1})^T,$$

$$\boldsymbol{z}^{j+1} = \left(z_1^{j+1} - \frac{\tau^2}{2h^2}\mu_L(t_{j+1}), z_2^{j+1}, \dots z_{n-2}^{j+1}, z_{n-1}^{j+1} - \frac{\tau^2}{2h^2}\mu_R(t_{j+1})\right)^T,$$

solving which will give us the layer of values $\{u_i^{j+1}\}_{i=1}^{n-1}$.

Since we need two layers – $\boldsymbol{u}^j$ and $\boldsymbol{u}^{j-1}$ to find $\boldsymbol{u}^{j+1}$, we cannot use this procedure to find $\boldsymbol{u}^1$. In that case we proceed thus – we assume that $u(x,t)$ is differentiable on the interval $t \in [-\tau, 0]$ and use the initial condition

$$\psi(x_i) = u_t(x_i, 0) = \frac{u_i^1 - u_i^{-1}}{2\tau} + O(\tau^2),\ i = \overline{0, n}$$

to express u_i^{-1}. Then we substitute it into (6) and, noting that $u_i^0 = \varphi(x_i)$, we get

$$u_i^1 = \frac{\tau^2}{2h^2}(\varphi(x_{i+1}) + \varphi(x_{i-1})) + \left(1 - \frac{\tau^2}{h^2}\right)\varphi(x_i) + \tau\psi(x_i) - \frac{\tau^2}{2}\sin(\varphi(x_i)),$$

through which we can find $\boldsymbol{u}^1$ explicitly and move on to the implicit scheme.

3. Numerical Results

Equations (2–5) have been used as a source of initial and boundary conditions.

In order to verify the method, we have analyzed the approximation error. Measuring the error is done by either comparing the results with the original solution or by substituting the found results in the partial differential equation, using finite differences as approximations. We will use the notation e_i^j for the error at the point (x_i, t_j).

The errors are presented by the maximum recorded error

$$\max_{i=\overline{0,n}, j=\overline{0,m}} |e_i^j|$$

and by applying the root mean square error (RMSE)

$$\sqrt{\frac{1}{(n+1)(m+1)} \sum_{j=0}^{m} \sum_{i=0}^{n} (e_i^j)^2}.$$

The parameters for the initial and boundary conditions are chosen thus:

1. Travelling wave – $s = 2, c = 2$,
2. Two-soliton – $s = 2, c_1 = c_2 = 0$,
3. Soliton-antisoliton – $s = 2, c_1 = c_2 = 0$,
4. Breather – $s = 0.5, c_1 = c_2 = 0$.

A summary of the results can be found in Table 1.

	Case 1 $a = -5, b = 5, T = 5, n = 200, m = 400$ $h^2 + \tau^2 = 2.66\text{e}{-3}$			
	Solution		PDE	
	Max	RMSE	Max	RMSE
Travelling wave	1.21e−2	2.32e−3	4.42e−3	1.02e−3
Two-soliton	1.21e−2	2.32e−3	4.42e−3	1.02e−3
Soliton-Antisoliton	1.12e−2	2.17e−3	3.67e−3	1.08e−3
Breather	2.54e−4	6.97e−5	3.27e−5	1.01e−5
	Case 2 $a = -5, b = 5, T = 5, n = 2000, m = 4000$ $h^2 + \tau^2 = 2.66\text{e}{-5}$			
	Max	RMSE	Max	RMSE
Travelling wave	1.16e−4	1.62e−5	2.86e−5	7.21e−6
Two-soliton	1.21e−4	2.32e−5	4.47e−5	9.97e−6
Soliton-Antisoliton	1.1e−4	2.17e−5	3.73e−5	1.06e−5
Breather	2.54e−6	6.99e−7	3.36e−7	1.02e−7

Table 1. Numerical results

Plots of some of the found solutions in case 2 and their corresponding errors are presented in Figures 2–5.

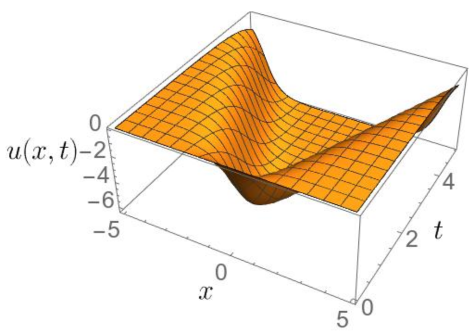

Figure 2. Soliton-antisoliton solution

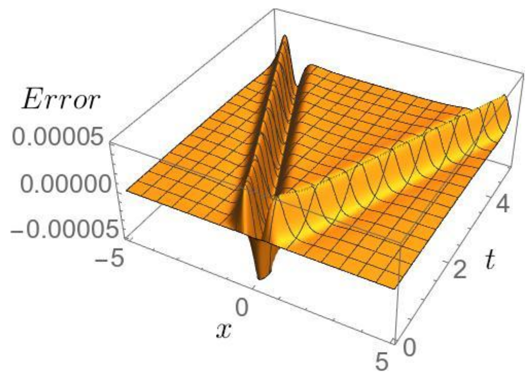

Figure 3. PDE error

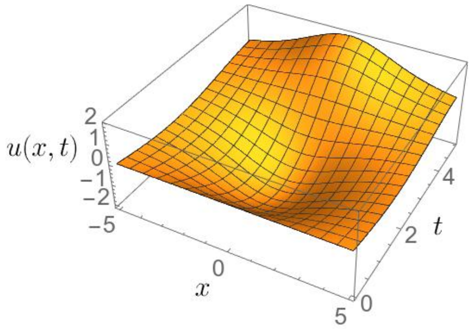

Figure 4. Breather solution

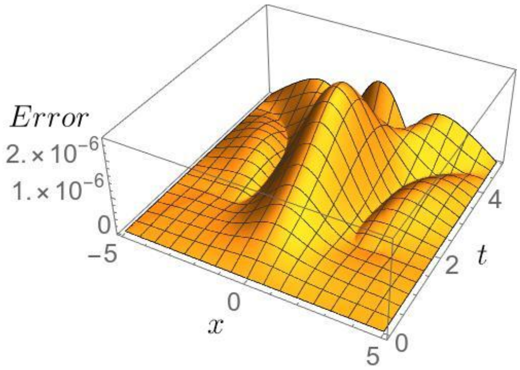

Figure 5. Solution error

As can be seen from Table 1 and Figures 2–5, the method performs according to the predicted order of accuracy, that is, second order in both x and t.

4. Conclusion

In this paper, we have utilized the Crank-Nicolson method to numerically solve the sine-Gordon equation. We have modified the classic scheme by considering an equation involving u_{tt} and by separating the linear and nonlinear parts and applying a linear interpolation on u_{xx}. As a results, we keep the implicit nature of the method, while removing the need to solve a nonlinear algebraic system of equations, instead solving a linear tridiagonal one using tridiagonal sweep.

As verification of the method, we have presented results from two cases with different step sizes and four sets of initial and boundary conditions. Two types of errors are utilized. Graphics of solutions and errors are presented.

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