EXISTENCE SOLUTIONS OF NONLINEAR FRACTIONAL INTEGRAL EQUATIONS WITH VARIABLE ORDER OF THE VOLTERRA TYPE WITH A NEW APPROACH

Kheireddine Benia, Zoubida Bouazza, Mohammed Said Souid

Abstract. *In this paper, we use a new approach to study the existence, uniqueness, and stability of solutions to a non-linear Volterra fractional integral equation with variable order under less restrictive presumptions. Finally, to illustrate the results of this work, we give an example which illustrate the applicability of our approach.*

Key words: Existence results for non-linear fractional integral equations of variable-order, Volterra integral equation, Fixed point theorem, Ulam-Hyers stability.

AMS (MOS) Subject Classifications: 26A33; 34A08; 34K37.

1. Introduction

Integral equation is an important field in the terrain of non-linear analysis methods, according to the many intervention in various researched problems. Numerous examined issues are often expressed in differential forms then converted into integral equations to do the existence of solution and to facilitate the resolution or make an approximation, see for example [2, 5, 6].

While many other research works on the existence of solutions to fractional constant order problems have been carried, the existence of solutions to variable-order problems is infrequently mentioned in the literature, and there have been only a few research papers on the stability of solutions. Lately, fractional calculus of variable-order have been considered in various phenomenons, such as: anomalous diffusion modelling, mechanical applications, multi-fractional Gaussian noises.

There are numerous papers that have study the existence of solutions of functional integral or differential equations of the fractional constant order. and we can see lately some papers investigating the existence of solutions with applying piecewise constant functions method, see for example [1, 2, 12].

In this paper, we will study the existence with new approach to replace piecewise constant functions method. As a result of investigating this intriguing special research topic, our findings are novel and notable.

We deal with a non-linear Volterra fractional integral equation with variable order,

$$x(t) = g(t) + \int_0^t \frac{(t-s)^{\kappa(s)-1}}{\Gamma(\kappa(s))} f(s, x(s)) ds, \qquad t \in V := [0, b], \tag{1}$$

where $0 \leq t \leq b < +\infty$, $0 < \kappa(t) \leq 1$ for all $s \in V$, Γ is the gamma function $g : V \longrightarrow \mathbb{R}$ is a continuous function and $f : V \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function.

2. Backgrounds

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

The following definitions is Riemann-Liouville fractional integral of variable-order for a function h. We consider the mapping $\kappa(t) : V \to (0, 1]$. Then, the left Riemann-Liouville fractional integral (RLFI) of variable-order $\kappa(t)$ for a function h is defined as [13, 14, 18]

$$I_{0^+}^{\kappa(t)} h(t) = \frac{1}{\Gamma(\kappa(t))} \int_0^t (t-s)^{\kappa(t)-1} h(s) ds, \qquad t > 0.$$

Provided that the right-hand side is pointwise defined

$$I_{0^+}^{\kappa(t)} h(t) = \int_0^t \frac{(t-s)^{\kappa(s)-1}}{\Gamma(\kappa(s))} h(s) ds, \qquad t > 0.$$

In the case when $\kappa(t)$ is a constant, then previous definitions coincides with the classical Riemann Liouville fractional integral of a constant order, see, e.g., [10, 13, 14].

Remark 2.1. *[21, 19] Note that, the semigroup property does not hold for arbitrary functions $\kappa(t)$, $v(t)$, i.e., in general*

$$I_{a^+}^{\kappa(t)} I_{a^+}^{v(t)} h(t) \neq I_{a^+}^{\kappa(t)+v(t)} h(t).$$

Lemma 2.1. *[20] If $\kappa \in C(V,(0,1])$, then:*
(a) The variable order fractional integral $I_{0^+}^{\kappa(t)}h(t)$ exists at any point on V for $h \in C_\sigma(V,\mathbb{R})$ where

$$C_\sigma(V,\mathbb{R}) = \{h(t) \in C(V,\mathbb{R}), t^\sigma h(t) \in C(V,\mathbb{R}), 0 \le \delta \le 1\}.$$

(b) $I_{0^+}^{\kappa(t)}h(t) \in C(V,\mathbb{R})$ for $h \in C(V,\mathbb{R})$.

In the proof of our main results we will also use the following fixed point theorem.

Theorem 2.1. *[10] (SFPT) Let W be a Banach Space and Q be a Convex, closed bounded and non-empty subset of W. If $\Upsilon : Q \longrightarrow Q$ is Completely Continuous map, then Υ has at least one Fixed point in Q.*

We give now the adaptive definition of the stability in Ulam–Hyers-Rassias sense.

Definition 2.1. *The Equation (1) is Ulam–Hyers-Rassias stable if there exists $C_f > 0$ such that for any $\epsilon > 0$ and for every solution $z \in C(V,R)$ of the inequality*

$$\left| z(t) - g(t) - \int_0^t \frac{(t-s)^{\kappa(s)-1}}{\Gamma(\kappa(s))} f(s,z(s))ds \right| \le \epsilon, \quad t \in V \tag{2}$$

there exists a solution $y \in C(V,R)$ of Equation (1) with

$$|z(t) - y(t)| \le c_f \epsilon, \quad t \in V.$$

3. Existence Solutions

Throughout the remainder portion of the study, we need the following assumptions:

(A1) $\kappa : V \to (0,\kappa']$ is continuous function, such that $0 < \kappa(t) \le \kappa' \le 1$.

(A2) The function $t^\sigma f$ is a continuous function on $V \times \mathbb{R}$ and there exist constants $0 \le \sigma < \min\limits_{0 \le t \le b} |\kappa(t)|$, $D > 0$ such that

$$t^\sigma |f(t,y) - f(t,\bar{y})| \le D|y - \bar{y}|,$$

for all $t \in V$ and $y,\ \bar{y} \in \mathbb{R}$.

Remark 3.1. *1) It follows from the continuity of compose functions that* $\Gamma(\kappa(t))$ *is continuous on* $[0, b]$*, when* κ *satisfies assumption condition (A1).*

Remark 3.2. *[19] 2) By the continuity of the function* $\kappa(t)$*, we let* $\kappa^* = \min_{0\leq t\leq b} |\kappa(t)|$*, thus for* $0 \leq t \leq b$*, we have*

$$b^{\kappa(t)-1} \leq 1, \text{ if } 1 \leq b \leq \infty,$$

or

$$b^{\kappa(t)-1} \leq b^{\kappa^*-1}, \text{ if } 0 < b \leq 1.$$

Thus for $b > 0$*, we conclude that*

$$b^{\kappa(t)-1} \leq \max\left\{1, b^{\kappa^*-1}\right\} = b^*.$$

Now, we will study the non linear fractional integral of Volterra types with variable order.

Theorem 3.1. *Let conditions (A1) and (A2) hold. If*

$$\frac{\Gamma(\kappa^\star)\Gamma(1-\sigma)b^\star D\ b^{1-\sigma}}{\Gamma(\kappa')\Gamma(1-\sigma+\kappa^\star)} < 1, \tag{3}$$

then the (1) has a unique solution in $C(V, \mathbb{R})$*.*

Proof: Consider the operator $\mathcal{V} : C(V, \mathbb{R}) \longrightarrow C(V, \mathbb{R})$ defined by

$$\mathcal{V}(y)(t) = g(t) + \int_0^t \frac{(t-s)^{\kappa(s)-1}}{\Gamma(\kappa(s))} f(s, y(s))ds.$$

To show that $\mathcal{V}$ admits a unique Fixed Point, it suffices to show that $\mathcal{V}$ is a contraction. Hence, by the Banach Contraction Principe, $\mathcal{V}$ has a unique Fixed Point $y \in C(V, \mathbb{R})$, which is a unique solution of the (1).

□

The second result validates the existence of the solutions, using Shauder Fixed Point Theorem.

Theorem 3.2. *Let conditions (A1) and (A2) hold. If*

$$\frac{r}{g^* + \dfrac{\left([Dr + f^\star]\, b^\star\right)\Gamma(\kappa^\star)\Gamma(1-\sigma)}{\Gamma(\kappa')\Gamma(1-\sigma+k^\star)} b^{1-\sigma}} \geq 1, \tag{4}$$

where $f^* = \sup_{t\in V} t^{\sigma}|f(t,0)|$, *then the equation* (1) *has at least one solution in* $C(V,\mathbb{R})$.

Proof: Define the set

$$B_r = \{y \in C(V,\mathbb{R}) : \|y\| \leq r\}.$$

Clearly, B_r is nonempty, bounded, closed and convex subset of $C(V,\mathbb{R})$.

The proof will be presented in three steps.

Step 1, Claim: $\mathcal{V}$ is continuous.

Step 2, Claim: $\mathcal{V}(B_r) \subseteq B_r$.

Step 3, Claim: $\mathcal{V}$ is compact. In order to show that $\mathcal{V}$ is compact, we demonstrate $\mathcal{V}(B_r)$ is relatively compact. By Step 2, we have that $\mathcal{V}(B_r)$ is uniformly bounded or $\mathcal{V}(B_r) = \{\mathcal{V}(y) : y \in B_r\} \subseteq B_r$. Thus, for each $y \in B_r$ we have $\|\mathcal{V}(y)\| \leq r$ which means that $\mathcal{V}(B_r)$ is bounded. It remains to indicate that $\mathcal{V}(B_r)$ is equicontinuous.

Hence, all conditions of the fixed point Theorem are satisfied and thus, Equation (1) has at least one solution $y \in B_r$. Since $B_r \subset C(V,\mathbb{R})$, the assertion of Theorem is proved.

$\square$

4. Ulam Hyers stability

In this section, we study the Ulam-Hyers stability for solutions to the problem.

Theorem 4.1. *Let the conditions of Theorem 3.2 be satisfied. Then, the integral Equation* (1) *is Ulam–Hyers stable.*

5. Example

In this example, we deal with the following integral equation of the Volterra type of fractional variable order.

Consider the following fractional problem

$$y(t) = \frac{1}{e^t+1} + \int_0^t \frac{(t-s)^{\frac{2}{7}s+\frac{3}{5}-1}}{\Gamma(\frac{2}{7}s+\frac{3}{5})} s^{-\frac{2}{7}} \frac{y(s)+1}{5} ds, \qquad t \in V := [0,1]. \qquad (5)$$

By identification we have $\kappa(t) = \dfrac{2}{7}t + \dfrac{3}{5}$, is a continuous function on V, and satisfies

$$\frac{3}{5} \le \kappa(t) \le \frac{31}{35}.$$

Also

$$f(t,y) = t^{-\frac{2}{7}}\left(1 + \frac{y+1}{5}\right),$$

is an continuous function on $(0,1] \times \mathbb{R}$. Clearly for $t \in [0,1]$, we have which implies that condition (A1) holds. Further, we have

$$\begin{aligned} t^{\frac{2}{7}}|f(t,y_1) - f(t,y_2)| &= \left| t^{\frac{2}{7}}\left[t^{-\frac{2}{7}}\left(1 + \frac{y_1+1}{5}\right) - t^{-\frac{2}{7}}\left(1 + \frac{y_2+1}{5}\right)\right]\right| \\ &\le \left|\frac{y_1+1}{5} - \frac{y_2+1}{5}\right| \\ &\le \frac{1}{5}|y_1 - y_2|. \end{aligned}$$

Hence condition (A2) holds with $\sigma = \frac{2}{7}$ and $D = \frac{1}{5}$. For the purpose of verifying (3), it is clear that

$$\frac{r}{1 + \frac{\Gamma(\frac{3}{5})\Gamma(\frac{5}{7})}{5\Gamma(\frac{31}{35})\Gamma(\frac{46}{35})}r + \frac{6\Gamma(\frac{3}{5})\Gamma(\frac{5}{7})}{5\Gamma(\frac{31}{35})\Gamma(\frac{46}{35})}} \ge 1,$$

is satisfied for each $r > 5.5$. Hence, condition (4). By Theorem 3.2, problem (5) has a unique solution.

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Kheireddine Benia[1], Zoubida Bouazza[2], Mohammed Said Souid[3]
[1] Departement of Mathematics, Ibn khaldoun university,
Tiaret, Algeria, kheireddine.benia@univ-tiaret.dz
[2] Department of informatics, University of Tiaret,
Algeria, bouazza3@yahoo.fr
[3] Department of Economic Sciences, Ibn khaldoun university,
Tiaret, Algeria
Corresponding author: souimed2008@yahoo.com